

THE ANGULAR MOMENTUM-ENERGY SPACE

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Abstract. In this paper we shall define and study the angular momentum-energy space for the classical problem of plane-motions of a particle situated in a potential field of a central force. We shall present the angular momentum-energy space for some important cases.

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1. Introduction

The angular momentum-energy states are used in Classical Mechanics to construct a mathematical model for the plane motions of a particle in a potential field of a central force (see [1], [4] or [5]).

In Astrophysics appears an equation in the angular momentum-energy space describing the stellar distribution around a black hole (see [2]).

In General Relativity a mathematical model of the motions of a particle use the concepts of energy and angular momentum (see [3]).

The objectives of this study are:

- to present the classical concepts of angular momentum and energy;
- to study the Angular Momentum-Energy Space and the Angular Momentum-Energy Space which are corresponding to the uniform rotations;
- to present the Angular Momentum-Energy Space and Angular Momentum-Energy Space which are corresponding to the uniform rotations for some particular force fields.

2. The Movements of a Particle in a Potential Field of a Central Force

We consider a particle situated in a potential field of central force. In the Newtonian Mechanics it is known that a trajectory is contained in a

plane which contains the center of the force. We study the case in which the trajectories are contained in a fixed plane passing through *center* O . We denote by \vec{r} the *radius vector* of the particle, r the *modulus of the radius vector* and $U_*(r)$ the *force function*. In this case the second law of Newton has the form:

$$m\ddot{\vec{r}} = -U'_*(r)\frac{\vec{r}}{r} \quad (1)$$

Projecting this equation on the natural base of the *polar coordinates* (r, φ) we have:

$$\begin{cases} \dot{r^2\dot{\varphi}} = 0 \\ \ddot{r} - r\dot{\varphi}^2 + U'(r) = 0 \end{cases} \quad (2)$$

where $U(r) = U_*(r)/m$ is the *force function per unit mass*.

We denote by J_* the *angular momentum*, E_* the *total energy*, $J = J_*/m$ the *angular momentum per unit mass* and $E = E_*/m$ the *total energy per unit mass*.

An other mathematical model of the motions is obtained using the conservation laws of the angular momentum and total energy. We have:

$$\begin{cases} r^2\dot{\varphi} = J \\ \frac{\dot{r}^2}{2} + \frac{J^2}{2r^2} + U(r) = E \end{cases} \quad (3)$$

Theorem 2.1. i) If (r, φ) is a solution of (2) and it is not an uniform rotation, then it exists $(J, E) \in \mathbf{R}^2$ such that (r, φ) is a solution of (3).
 ii) If $(J, E) \in \mathbf{R}^2$, (r, φ) is a solution of (3) and it is not an uniform rotation, then (r, φ) is a solution of (2).
 iii) If $r_0 > 0$, then it exists an uniform rotation (r_0, φ) solution of (2) and (3) if and only if:

$$\begin{cases} J^2 = r_0^3 U'(r_0) \\ E = U(r_0) + \frac{r_0 U'(r_0)}{2} \end{cases} \quad (4)$$

Proof. The propositions i) and ii) are classical results.

iii) Let $(J, E) \in \mathbf{R}^2$ and $r_0 > 0$ such that the relation (4) is true. The uniform rotation $(r_0, \frac{Jt}{r_0^2})$ is a solution of (2) and (3). Let $(J, E) \in \mathbf{R}^2$, $r_0 > 0$ and (r_0, φ) an uniform rotation which is a solution of (2) and (3) then we have:

$$\dot{\varphi} = \frac{J}{r_0^2}, \quad r_0\dot{\varphi}^2 - U'(r_0) = 0, \quad \frac{J^2}{2r_0^2} + U(r_0) = E. \quad (5)$$

We introduce $(5)_1$ in $(5)_2$ and we obtain $(4)_1$. The relation $(4)_2$ is obtained using $(4)_1$ and $(5)_3$.

Hypothesis: *In this paper we suppose that a force function per unit mass is a function $U \in C^1((0, \infty), \mathbf{R})$.*

Remark 2.1. The most important is the case of an attractive force field which is characterized by a force function with the property $U' > 0$.

In this paper we use the following notations: *the effective force function per unit mass:*

$$\mathbf{V}_J^U(r) = \frac{J^2}{2r^2} + U(r) \quad (6)$$

the effective angular momentum per unit mass:

$$\mathbf{W}_E^U(r) = 2r^2(E - U(r)) \quad (7)$$

where U is a force function per unit mass, E the total energy per unit mass and J the angular momentum per unit mass. We have $\mathbf{V}_J^U, \mathbf{W}_E^U \in C^1((0, \infty), \mathbf{R})$.

It is easy to see:

Proposition 2.1. *If (r, φ) is a solution of (3), then for all time-moments we have:*

$$\mathbf{V}_J^U(r(t)) \leq E \quad (8)$$

$$\mathbf{W}_E^U(r(t)) \geq J^2 \quad (9)$$

The most important notions for the paper are presented in the next considerations.

Definition 2.1. Let U a force function per unit mass; $(J, E) \in \mathbf{R}^2$ is an *angular momentum-energy state* if it exists a motion of the particle with J the angular momentum per unit mass and E the total energy per unit mass. The *Angular Momentum-Energy Space* \mathbf{S}_U is the set of the angular momentum-energy states.

Remark 2.2. $(J, E) \in \mathbf{R}^2$ is an angular momentum-energy state if and only if it exists a solution (r, φ) of (2) and (3).

Remark 2.3. An angular momentum-energy state (J, E) is corresponding to an uniform rotation if exists an uniform rotation of the particle with J

the angular momentum per unit mass and E the total energy per unit mass. We denote by $\mathbf{S}_U^{u,r}$ the set of the angular momentum-energy states which are corresponding to the uniform rotations.

3. The Angular Momentum-Energy Space

3.1. The set of the angular momentum-energy states which are corresponding to the uniform rotations. This set is characterized by the theorem 1, we have:

$$\mathbf{S}_U^{u,r} = \{(J, E) \mid \exists s > 0 \ J^2 = s^3 U'(s) \text{ and } E = U(s) + \frac{sU'(s)}{2}\} \quad (10)$$

It is easy to see that we have the following characterizations of the set of angular momentum-energy states which are corresponding to the uniform rotations:

$$\mathbf{S}_U^{u,r} = \{(J, E) \mid \exists s > 0 \ (\mathbf{V}_J^U)'(s) = 0 \text{ and } E = \mathbf{V}_J^U(s)\} \quad (11)$$

$$\mathbf{S}_U^{u,r} = \{(J, E) \mid \exists s > 0 \ J^2 = \mathbf{W}_E^U(s) \text{ and } (\mathbf{W}_E^U)'(s) = 0\} \quad (12)$$

We present some interesting properties of the set $\mathbf{S}_U^{u,r}$.

- Proposition 3.1.** i) If $(J, E) \in \mathbf{S}_U^{u,r}$, then $(-J, E) \in \mathbf{S}_U^{u,r}$.
 ii) If $r_0 > 0$, then it exists an uniform rotation $r(t) = r_0$ if and only if $U'(r_0) \geq 0$.
 iii) If $\min_{r>0} \mathbf{V}_J^U(r) \in \mathbf{R}$ and $E = \min_{r>0} \mathbf{V}_J^U(r)$, then $(J, E) \in \mathbf{S}_U^{u,r}$.
 iv) If $\max_{r>0} \mathbf{W}_E^U(r) \in \mathbf{R}_+$ and $J = \sqrt{\max_{r>0} \mathbf{W}_E^U(r)}$ then $(J, E) \in \mathbf{S}_U^{u,r}$.

Proof. The first and second results are consequences of the characterization (7) of the set $\mathbf{S}_U^{u,r}$.

iii) In our hypotheses it exists $r_0 > 0$ such that $E = \mathbf{V}_J^U(r_0)$. According to Fermat theorem we have $(\mathbf{V}_J^U)'(r_0) = 0$. It is easy to see that the relations (4) are verified and $(J, E) \in \mathbf{S}_U^{u,r}$.

The proof of iv) is analogue with the demonstration of iii).

3.2. The properties of the Angular Momentum-Energy Space.

Firstly we present a characterization of the Angular Momentum-Energy Space using the properties of the force function per unit mass U .

Theorem 3.1. *The Angular Momentum-Energy Space is characterized by:*

$$\mathbf{S}_U = \{(J, E) \mid E > \inf_{r>0} \mathbf{V}_J^U(r) \text{ or } E = \min_{r>0} \mathbf{V}_J^U(r)\} \quad (13)$$

and

$$\mathbf{S}_U = \{(J, E) \mid J^2 < \sup_{r>0} \mathbf{W}_E^U(r) \text{ or } J^2 = \max_{r>0} \mathbf{W}_E^U(r)\} \quad (14)$$

Proof. We suppose that $(J, E) \in \mathbf{S}_U$. It exists (r, φ) a solution of (2) and (3). It is easy to see that $E \geq \inf_{r>0} \mathbf{V}_J^U(r)$ and $J^2 \leq \sup_{r>0} \mathbf{W}_E^U(r)$.

If $E = \inf_{r>0} \mathbf{V}_J^U(r)$, then it exists $r_0 > 0$ such that $E = \mathbf{V}_J^U(r_0) = \min_{r>0} \mathbf{V}_J^U(r)$. We deduce that:

$$\mathbf{S}_U \subset \{(J, E) \mid E > \inf_{r>0} \mathbf{V}_J^U(r) \text{ or } E = \min_{r>0} \mathbf{V}_J^U(r)\}$$

If $J^2 = \sup_{r>0} \mathbf{W}_E^U(r)$, then it exists $r_0 > 0$ such that $J^2 = \mathbf{W}_E^U(r_0) = \max_{r>0} \mathbf{W}_E^U(r)$. We deduce that:

$$\mathbf{S}_U \subset \{(J, E) \mid J^2 < \sup_{r>0} \mathbf{W}_E^U(r) \text{ or } J^2 = \max_{r>0} \mathbf{W}_E^U(r)\}$$

Let $(J, E) \in \{(J, E) \mid E > \inf_{r>0} \mathbf{V}_J^U(r) \text{ or } E = \min_{r>0} \mathbf{V}_J^U(r)\}$. If $E > \inf_{r>0} \mathbf{V}_J^U(r)$, then it exists $r_0 > 0$ such that $E > \mathbf{V}_J^U(r_0)$. We consider the Cauchy problem of differential equations:

$$\dot{\varphi} = \frac{J}{r^2}, \quad \dot{r} = 2\sqrt{E - \mathbf{V}_J^U(r)}, \quad \varphi(0) = \frac{J}{r_0^2}, \quad r(0) = r_0.$$

According to the Cauchy-Lipschitz Theorem the Cauchy problem has a solution (r, φ) . This solution is not an uniform rotation and it is a solution of the system (3). Using the Theorem 2.1 we deduce that (r, φ) is a solution of (2) and we conclude that:

$$\mathbf{S}_U \supset \{(J, E) \mid E > \inf_{r>0} \mathbf{V}_J^U(r) \text{ or } E = \min_{r>0} \mathbf{V}_J^U(r)\}$$

Let $(J, E) \in \{(J, E) \mid J^2 < \sup_{r>0} \mathbf{W}_E^U(r) \text{ or } J^2 = \max_{r>0} \mathbf{W}_E^U(r)\}$. If $J^2 < \sup_{r>0} \mathbf{W}_E^U(r)$, then it exists $r_0 > 0$ such that $J^2 < \mathbf{W}_E^U(r_0)$. We consider the Cauchy problem of differential equations:

$$\dot{\varphi} = \frac{J}{2u}, \quad \dot{u} = \sqrt{\mathbf{W}_E^U(\sqrt{2u}) - J^2}, \quad \varphi(0) = \frac{J}{r_0^2}, \quad u(0) = \frac{r_0^2}{2}$$

According to the Cauchy-Lipschitz Theorem the Cauchy problem has a solution (u, φ) . In this situation $(r, \varphi) = (\sqrt{2u}, \varphi)$ is a solution of the system (3). We obtain:

$$\mathbf{S}_U \supset \{(J, E) \mid J^2 < \sup_{r>0} \mathbf{W}_E^U(r) \text{ or } J^2 = \max_{r>0} \mathbf{W}_E^U(r)\}$$

We present some properties of the Angular Momentum-Energy Space.

Theorem 3.2. *i) If $(J, E) \in \mathbf{S}_U$, then $(-J, E) \in \mathbf{S}_U$.*

ii) If $k \in \mathbf{R}$, then $\mathbf{S}_{U+k} = \mathbf{S}_U + (0, k)$.

iii) Let U_1, U_2 two force functions, if $U_1 \leq U_2$, then $\mathbf{S}_{U_2} \subset \mathbf{S}_{U_1}$.

Proof. i) We observe that $\mathbf{V}_J^U = \mathbf{V}_{-J}^U$ and one obtains the affirmation.

ii) The result is an immediate consequence of the relation $\mathbf{V}_J^{U+k} = \mathbf{V}_J^U + k$.

iii) We have $\mathbf{V}_J^{U_1}(r) \leq \mathbf{V}_J^{U_2}(r) \forall r > 0$ which implies easily our proposition.

Remark 3.1. $U_1 \leq U_2 \Leftrightarrow \forall r > 0$ we have $U_1(r) \leq U_2(r)$.

Finally we study the conditions of the force function per unit mass U such that the Angular Momentum-Energy Space is the entire \mathbf{R}^2 .

Theorem 3.3. *The next affirmations are equivalents:*

i) $\mathbf{S}_U = \mathbf{R}^2$.

ii) $\liminf_{r \rightarrow 0} r^2 U(r) = -\infty$ or $\liminf_{r \rightarrow \infty} U(r) = -\infty$.

Proof. Firstly we suppose that $\mathbf{S}_U = \mathbf{R}^2$, $\liminf_{r \rightarrow 0} r^2 U(r) > -\infty$ and $\liminf_{r \rightarrow \infty} U(r) > -\infty$.

It exists $0 < r^* < r^{**}$ and $k^*, k^{**} \in \mathbf{R}_+^*$ such that, if $r \in (0, r^*)$, then $U(r) > -\frac{k^*}{r^2}$ and if $r > r^{**}$, then $U(r) > -k^{**}$. U is a continuous function and $[r^*, r^{**}]$ is a compact interval, there exists $k^{***} > 0$ such that $U(r) > k^{***}$ for all $r \in [r^*, r^{**}]$. We introduce $\tilde{k} = \max\{k^*, k^{***}r^{*2}, k^{**}r^{**2}\} > 0$ and we have $U(r) \geq -\frac{\tilde{k}}{r^2}$ for all $r > 0$. Using the Theorem 3.2. one obtains that $\mathbf{S}_U \subset \mathbf{S}_{-\frac{\tilde{k}}{r^2}}$. We known that $\mathbf{S}_{-\frac{\tilde{k}}{r^2}} \neq \mathbf{R}^2$ (see the §4.3.) and we deduce that $\mathbf{S}_U \neq \mathbf{R}^2$, but this result is a contradiction.

If the affirmation ii) is true, then $\inf_{r>0} \mathbf{V}_J^U(r) = -\infty$ for all $J \in \mathbf{R}$. According to Theorem 3.1. we obtain that the affirmation i) is true.

4. Particular cases of Angular Momentum-Energy Space

4.1. An Isolated Particle; $U = 0$.

$$\mathbf{S}_0 = \{(J, E) \in \mathbf{R}^2 / E > 0\} \cup \{(0, 0)\}, \quad \mathbf{S}_0^{u.r} = \{(0, 0)\} \quad (15)$$

In this case an uniform rotation is an equilibrium point. The angular momentum-energy state $(0, 0)$ is corresponding to all uniform rotations (equilibrium points).

Remark 4.1. Let $k \in \mathbf{R}$. Using the theorem 3 we obtain:

$$\mathbf{S}_k = \{(J, E) / E > k\} \cup \{(0, 0)\} \quad (16)$$

4.2. Particle in a Gravitational Force Field, $U = -\frac{k}{r}$. We suppose that the gravitational force is an attraction force ($k > 0$). In this case we have:

$$\mathbf{S}_{-\frac{k}{r}} = \{(J, E) / EJ^2 \geq -\frac{k^2}{2}\}, \quad \mathbf{S}_{-\frac{k}{r}}^{u.r} = \{(J, E) / EJ^2 = -\frac{k^2}{2}\} \quad (17)$$

4.3. $U = -\frac{k}{r^2}$ with $k > 0$. We have:

$$\mathbf{S}_{-\frac{k}{r^2}} = S_1 \cup S_2 \cup S_3, \quad \mathbf{S}_{-\frac{k}{r^2}}^{u.r} = \{(-\sqrt{2k}, 0), (\sqrt{2k}, 0)\} \quad (18)$$

where:

$$\begin{cases} S_1 = \{(J, E) / (J^2 > 2k \text{ and } E > 0)\} \\ S_2 = \{(J, E) / J^2 = 2k \text{ and } E \geq 0\} \\ S_3 = \{(J, E) / J^2 < 2k \text{ and } E \in \mathbf{R}\} \end{cases}$$

4.4. Particle in a Hooke Force Field, $U = \frac{k}{2}r^2$ with $k > 0$. We have:

$$\mathbf{S}_{\frac{k}{2}r^2} = \{(J, E) / E \geq \sqrt{k}|J|\} - \{(0, 0)\}, \quad \mathbf{S}_{\frac{k}{2}r^2}^{u.r} = \{(J, E) / E = \sqrt{k}|J|\} - \{(0, 0)\} \quad (19)$$

Remark 4.2. For us U is not defined for $r = 0$; this is the reason for which $(0, 0)$ is not a angular momentum-energy state.

4.5. Particle in a Repulsive Elastic force Field, $U = -\frac{k}{2}r^2$ with $k > 0$.

In this case:

$$\mathbf{S}_{-\frac{k}{2}r^2} = \mathbf{R}^2, \quad \mathbf{S}_{-\frac{k}{2}r^2}^{u.r} = \emptyset \quad (20)$$

4.6. $U = -\frac{k}{r} - \frac{q}{r^2}$ with $k > 0$ and $q > 0$.

$$\mathbf{S}_{-\frac{k}{r}-\frac{q}{r^2}} = \{(J, E) / J^2 \leq 2q \text{ or } (J^2 > 2q \text{ and } E(J^2 - 2q) \geq -\frac{k^2}{2})\} \quad (21)$$

$$\mathbf{S}_{-\frac{k}{r}-\frac{q}{r^2}}^{u.r} = \{(J, E) / E < 0 \text{ and } E(J^2 - 2q) = -\frac{k^2}{2}\} \quad (22)$$

4.7. $U = -\frac{k}{r^{2n}}$ with $k > 0$ and $n > 0$. If $n > 1$, then we have:

$$\mathbf{S}_{-\frac{k}{r^{2n}}} = \mathbf{R}^2, \quad \mathbf{S}_{-\frac{k}{r^{2n}}}^{u.r} = \{(J, E) / EJ^{\frac{2n}{1-n}} = (n-1)(2n)^{\frac{n}{1-n}}k^{\frac{1}{1-n}}\} \quad (23)$$

The case $n = 1$ is studied in the §4.3.

If $n \in (0, 1)$ then:

$$\mathbf{S}_{-\frac{k}{r^{2n}}} = \{(J, E) / EJ^{\frac{2n}{1-n}} \geq -(1-n)(2n)^{\frac{n}{1-n}}k^{\frac{1}{1-n}}\} \quad (24)$$

$$\mathbf{S}_{-\frac{k}{r^{2n}}}^{u.r} = \{(J, E) / EJ^{\frac{2n}{1-n}} = -(1-n)(2n)^{\frac{n}{1-n}}k^{\frac{1}{1-n}}\} \quad (25)$$

4.8. $U = q \sin \frac{1}{r}$ with $q > 0$. This case is interesting for theoretical reasons. Using the theorem of characterization of the angular momentum-energy Space we obtain:

$$\mathbf{S}_{q \sin \frac{1}{r}} = \{(J, E) / E > -q\} \cup \{(0, -q)\} \quad (26)$$

The set $D_{q \sin \frac{1}{r}}$ of the distances r_0 for which exists an uniform rotations with $r(t) = r_0$ is described by the formula:

$$D_{q \sin \frac{1}{r}} = \cup_{k \in \mathbf{N}} \left[\frac{2}{(4k+3)\pi}, \frac{2}{(4k+1)\pi} \right] \quad (27)$$

If $r_0 \in \{\frac{2}{(2k+1)\pi} / k \in \mathbf{N}\}$, then at the distance r_0 the particle can have an equilibrium state. All equilibrium states have an angular momentum-energy state in the set $\{(0, q), (0, -q)\}$.

The set of the angular momentum-energy states which are corresponding to the uniform rotations is:

$$\mathbf{S}_{q \sin \frac{1}{r}}^{u,r} = \{(J, E) / \exists s > 0 \ J^2 = -qs \cos \frac{1}{s} \text{ and } E = q \sin \frac{1}{s} - \frac{q}{2s} \cos \frac{1}{s}\} \quad (28)$$

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